Chapter 10

Horizontal, Spiral and Vertical Curves

Topics to be covered

- Types of Horizontal Curves
- Deflection Angles, Chord and Offset Calculations
- Compound and Reverse Curves
- Spiral Curves
- Vertical Curves
- Geometric Properties of Vertical Curves
- High and Low Points on Vertical Curves
- Asymmetrical Vertical Curves

16 Sample Problems with Detailed Solutions

10 Supplemental Practice Problems with Detailed Solutions
Chapter 10- Horizontal, Spiral and Vertical Curves

10-1 INTRODUCTION

Horizontal curves may be simple, compound, reverse, or spiral. Compound and reverse curves are treated as a combination of two or more simple curves, whereas the spiral curve is based on a varying radius.

Curves of short radius (usually less than one tape length) can be established by holding one end of the tape at the center of the circle and swinging the tape in an arc, marking as many points as may be desired. As the radius and length of curve increases, the tape becomes impractical and the surveyor must use other methods. The common method is to measure angles and straight-line sight distances by which selected points, known as stations, may be located on the circumference of the arc.

![Horizontal Curves Diagram](image)

**Figure 10.1** Types of Horizontal Curves

10-2 TYPES OF HORIZONTAL CURVES

**Table 10-1** Types of Horizontal Curves

<table>
<thead>
<tr>
<th>Simple Circular</th>
<th>Compound</th>
<th>Reverse</th>
<th>Spiral</th>
</tr>
</thead>
<tbody>
<tr>
<td>The simple curve is an arc of a circle. The radius of the circle determines the sharpness or flatness of the curve. The larger the radius, the flatter the curve. This type of curve is the most often used.</td>
<td>Frequently the terrain will necessitate the use of a compound curve. This curve normally consists of two simple curves joined together, but curving in opposite directions. For safety reasons, this curve is seldom used in highway construction as it would tend to send an automobile off the road.</td>
<td>A reverse curve consists of two simple curves joined together, but curving in the same direction.</td>
<td>The spiral is a curve which has a varying radius. It is used on railroads and some modern highways. Its purpose is to provide a transition from the tangent to a simple curve or between simple curves in a compound curve.</td>
</tr>
</tbody>
</table>
10-3 TERMINOLOGY OF HORIZONTAL CURVES

Following are the main elements of a simple curve; see Fig.10.2

1. **Point of intersection**: the point of intersection (PI) is the point where the back and forward tangents intersect.

2. **The radius (R)**: the radius of the circle of which the curve is an arc.

3. **The point of curvature**: the point of curvature (PC) is the point where the circular curve begins. The back tangent is tangent to the curve at this point.

4. **The point of tangency**: the point of tangency (PT) is the end of the curve. The forward tangent is tangent to the curve at this point.

   **Note**: The terms BC (Beginning of Curve) and EC (End of Curve) are referred to by some agencies as PC (point of curvature) and PT (point of tangency), and by others as TC (tangent to curve) and CT (curve to tangent).

5. **The length of curve (L)**: the distance from the PC to the PT measured along the curve.

6. **The tangent distance (T)**: the distance along the tangents from the PI to the PC or PT. These distances are equal on a simple curve.

7. **The central angle (Δ)**: the angle formed by two radii drawn from the center of the circle (O) to the PC or PT. The central angle is equal in value to the intersecting angle (Δ = I).

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**Figure 10.2** Terminology of Horizontal Curve
8. **Long chord:** The long chord (LC or C) is the chord from the PC to the PT.

9. **External distance:** The external distance (E) is the distance from the PI to the midpoint of the curve. The external distance bisects the interior angle at the PI.

10. **Middle ordinate:** The middle ordinate (M) is the distance from the midpoint of the curve to the midpoint of the long chord. The extension of the middle ordinate bisects the central angle.

11. **Degree of curve:** The degree of curve (D) defines the "sharpness" or "flatness" of the curve. There are two common definitions for degree of curve, as follows:

   **Table 10-2 Chord and Arc Definitions for Horizontal Curves**

<table>
<thead>
<tr>
<th>Chord Definition</th>
<th>Arc Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>The chord definition states that the degree of a curve is the angle formed by two radii drawn from the center of the circle to the ends of a chord 100 ft long. The chord definition is used primarily for civilian railroad construction and is used by the military for both roads and railroads.</td>
<td>The arc definition states that the degree of a curve is the angle formed by two radii drawn from the center of the circle to the ends of an arc 100 ft long. This definition is used primarily for highways and streets. Notice that the larger the degree of curve, the &quot;sharper&quot; the curve and the shorter the radius.</td>
</tr>
<tr>
<td>[ \sin\left(\frac{D}{2}\right) = \frac{50,\text{ft}}{R} ] (10-1)</td>
<td>[ D = \frac{(360^\circ)(100,\text{ft})}{2\pi R} = \frac{5729.58^\circ}{R} ] (10-2)</td>
</tr>
</tbody>
</table>

The sharpness of a curve is determined by the choice of the radius (R); large radius curves are relatively flat, whereas small radius curves are relatively sharp.
**10-4 GEOMETRY OF HORIZONTAL CIRCULAR CURVES**

**Tangent (BC to PI OR PI to EC)**: \[ T = R \tan \frac{\Delta}{2} \] (10-3)

**Long Chord (BC to B to EC)**: \[ C = 2R \sin \frac{\Delta}{2} = 2T \cos (\Delta/2) \] (10-4)

**Curve Length (BC to EC Along the Curve i.e. BC to A to EC)**:

\[ L = 2\pi R \frac{\Delta^\circ}{360^\circ} = R \Delta (\text{radians}) = (100 \text{ ft})(\frac{\Delta}{D}) \] (10-5)

**Middle Ordinate (A to B)**: \[ M = R (1 - \cos \frac{\Delta}{2}) = \frac{C}{2} \tan \frac{\Delta}{4} = E \cos \frac{\Delta}{2} \] (10-6)

**External Dist. (PI to A)**: \[ E = R \left[ \frac{1}{\cos(\Delta/2)} - 1 \right] = R (\sec \frac{\Delta}{2} - 1) \]

\[ = T \tan \frac{\Delta}{4} = R \tan \frac{\Delta}{2} \tan \frac{\Delta}{4} \] (10-7)

**Notes:**

1. \( \cos \frac{\Delta}{2} = \frac{R}{R + E} \quad \text{i.e.} \quad \Delta = 2 \cos^{-1} \left( \frac{R}{R + E} \right) \)

2. versed sine (vers) \( \rightarrow \) vers (\(\Delta/2\)) = 1 - \(\cos (\Delta/2)\)

3. external secant (exsec) \( \rightarrow \) exsec (\(\Delta/2\)) = \(\sec (\Delta/2) - 1\)

4. A common mistake is to determine the station of the “EC” by adding the “T” distance to the “PI”. Although the “EC” is physically a distance of “T” from the “PI”, the stationing (chainage) must reflect the fact that the centerline no longer goes through the “PI”. The centerline now takes the shorter distance “L” from the “BC” to the “EC”.

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Sample Problem 10-1: Horizontal Curve Chord, Middle Ordinate & External Distance

Given: $\Delta = 16^\circ 38'$, $R = 1000$ ft, PI Sta. @ 6 + 26.57

Find: BC and EC stations, length of chord (C), middle ordinate (M), and external distance (E)

Solution:

\[ T = R \tan \frac{\Delta}{2} = 1000 \tan 8.3167^\circ = 146.18 \text{ ft} \]

\[ L = 2\pi R \frac{\Delta (\text{deg.})}{360} = R\Delta (\text{radians}) = (100 \text{ ft}) \left( \frac{\Delta}{D} \right) \]

\[ = 2\pi \times 1000 \times \frac{16.6333}{360} = 290.31 \text{ ft} \]

PI at 6 + 26.57

$-T$ 1 + 46.18

BC = 4 + 80.39 \Rightarrow

+ L 2 + 90.31

EC = 7 + 70.70 \Rightarrow

\[ C = 2R \sin \frac{\Delta}{2} = 2T \cos (\Delta/2) = 2 \times 1000 \times \sin 8.3167^\circ = 289.29 \text{ ft} \]

\[ M = R(1 - \cos \frac{\Delta}{2}) = \frac{1}{2} CTan \frac{\Delta}{2} = 1000(1 - \cos 8.3167^\circ) = 10.52 \text{ ft} \]

\[ E = R \left( \frac{1}{\cos (\Delta/2)} - 1 \right) = R \frac{\Delta}{2} Tan \frac{\Delta}{4} = 1000(\sec 8.3167^\circ - 1) = 10.63 \text{ ft} \]
### Sample Problem 10-2: Using Degree of Horizontal Curves

**Given:** \( \Delta = 11^\circ 21'35'' \), degree of curve \( D = 6^\circ \), PI Sta. @ 14 + 87.33

**Find:** BC and EC stations

#### Solution:

\[
D = \frac{(360^\circ)(100\text{ ft})}{2\pi R} = \frac{5729.58^\circ}{R} \quad (10-2)
\]

\[\Rightarrow R = \frac{5729.58}{D} = 954.93 \text{ ft}\]

\[T = RT\tan\frac{\Delta}{2} = 954.93 \tan 5.6799^\circ = 94.98 \text{ ft}\]

\[L = 2\pi R \frac{\Delta(\text{deg.})}{360} = R\Delta(\text{radians}) = (100 \text{ ft})\left(\frac{\Delta}{D}\right) = \frac{100 \times 11.3598}{6} = 189.33 \text{ ft}\]

- PI at 14 + 87.33
- \(-T\) 00 + 94.98
- BC = 13 + 92.35 \(\Leftarrow\)
- + L 01 + 89.33
- EC = 15 + 81.68 \(\Leftarrow\)

**Note:** A common mistake is to determine the station of the EC by adding the T distance to the PI station.
DEFLECTION ANGLES, CENTRAL ANGLE, & CHORD CALCUATIONS

The deflection angle is defined as the angle between the tangent and a chord. The following two rules apply for the deflection angles for circular curves:

**Rule 1:** The deflection angle between a tangent and a chord is half the central angle subtended by the arc i.e. the angle between the tangent “BC-PI” and the chord “PC-A” is $\frac{1}{2}$ the central angle “BC-O-A” i.e. $\alpha$ & $2\alpha$

**Rule 2:** The angle between two chords is $\frac{1}{2}$ the central angle subtended by the arc between the two chords i.e. the angle “A-BC-B” is $\frac{1}{2}$ the central angle “A-O-B” i.e. $\beta$ & $2\beta$

\[
deflection\ angle = \left(\frac{\text{arc length}}{L}\right)\left(\frac{\Delta}{2}\right)
\]

\[
deflection\ angle = \left(\frac{\text{arc length}}{L}\right)\left(\frac{\Delta}{2}\right)
\]

Chord Length (BC to A) = $2R \sin \alpha$

\[
\alpha = \frac{\text{arc length (BC to A)} \times 180^0}{2\pi R}
\]

\[
2\alpha = \frac{\text{arc length (BC to A)}}{\Delta} = \frac{L}{\Delta}
\]

Abbreviations:
- BC = Beginning of curve
- PC = Point of curvature
- TC = Tangent to curve
- EC = End of curve
- PT = Point of tangency
- CT = Curve to tangent

**Figure 10.3** Deflection and Central Angles Realtieship
10-13  GEOMETRIC PROPERTIES OF THE PARABOLA

1. The difference in elevation between the BVC and a point on the $g_1$ grade line at a distance $x$ units (feet or meters) is $g_1 x$ ($g_1$ is expressed as a decimal).

![Figure 10.14  Geometric of a Parabola](image)

2. The tangent offset between the grade line and the curve is given by $ax^2$, where $x$ is the horizontal distance from the BVC (PVC); that is, tangent offsets are proportional to the squares of the horizontal distances.

3. The elevation of the curve at distance $x$ from the BVC is given by:

$$ y = ax^2 + bx + c \quad (\text{general equation for a parabola}) \quad (10-24) $$

$$ y_x = y_{BVC} + g_1 x + \frac{rx^2}{2} \quad (10-25) $$

$$ r = \frac{g_2 - g_1}{L} \quad (10-26) $$

Where: $x$ = the distance from BVC to a point on the curve 
$r$ = rate of grade change per station

4. The grade lines ($g_1$ and $g_2$) intersect midway between the BVC and the EVC; that is, BVC to PVI = $\frac{1}{2} L$ = PVI to EVC. This is only true for symmetrical vertical curves.

5. The curve lies midway between the PVI and the midpoint of the chord; that is, $A-B = B-PVI = d_0$ which can be calculated as follows:

Either:

$$ d_0 = \frac{1}{2} \text{ (difference in elevation of PVI and mid-chord elevation)} $$

$$ = \frac{1}{2} \text{ (elevation of BVC + elevation of EVC)} $$
6. The slope $S$, in percentage, of the tangent to the curve at any point on the curve is given by the following formula:

$$S = g_1 - \frac{x(g_1 - g_2)}{L}$$

(10-28)

![Diagram of Crest and Sag Vertical Curves Terminology](image)

**Figure 10.15** Crest and Sag Vertical Curves Terminology

7. The distance $D$ in feet from Vertex to $P'$ is given as:

$$D = \frac{100(Y_H - Y_P')}{(g_1 - g_2)}$$

(10-29)

8. The distance between the curve and the grade line (tangent) “$d$” is given as

$$d = \text{offset} = \frac{rx^2}{2} = \frac{x^2(g_2 - g_1)}{200L}$$

(L curve length in feet)

(10-30)

### 10-14 HIGH AND LOW POINTS ON VERTICAL CURVES

The locations of the curves high and low points are important for drainage considerations; for example, on curbed streets catch basins must be installed precisely at the drainage low point. From equation (10-25), the slope $(dy/dx)$ is equaled to zero and solving for $X$:

$$g_1 + rx = 0$$

(10-31)
Figure 10.16  Low Point on a Sag Vertical Curve

\[ X = \frac{-g_1}{r} = \frac{-g_1L}{g_2 - g_1} = \frac{g_1L}{g_1 - g_2} \]  \hspace{1cm} (10-32)

Where \( X \) is the distance from BVC to the low or high points.
It should be noted that the distance \( X \) in the above two equations is different from distance \( x \) in equations 10-24 & 10-25.

**Sample Problem 10-13: Low point on a vertical curve**

**Given:** \( L = 300 \text{ ft}, g_1 = -3.2\%, g_2 = +1.8\%, \) PVI at 30 + 30, and elevation = 485.92

**Find:** Location of the low point and its elevation.

**Solution:**

\[ X = \frac{-g_1}{r} = \frac{-g_1L}{g_2 - g_1} = \frac{g_1L}{g_1 - g_2} = \frac{-(-3.2)(3)}{(1.8) - (-3.2)} = 1.92 \text{ Sta.} = 192.00 \text{ ft} \]

This means that the low point is located at a distance of 192.00 ft from BVC i.e. at Station = [(30 + 30.00) - (1+ 50.00)] + (1 + 92.00) = 30 + 72.00

**Remember:** All distances used to located a low or a high point or used to determine an elevation of a point on a vertical curve are measured from BVC.

\[ y' = y_{BVC} + g_1x + \frac{rx^2}{2} \]

\[ = [485.92 + (1.5)(3.2)] + (-3.2)(1.92) + \left( \frac{1.8 - (-3.2)}{3.00} \right) \left( \frac{1.92^2}{2} \right) = 487.65 \text{ ft @ Sta 30 + 72.00} \]